J. Fluid Mech. (1972), *vol.* 56, *part* 2, *pp.* 257–263
Printed in Great Britain

Simple waves on shear flows: similarity solutions

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(Received 26 April 1972)

A class of similarity solutions is derived for the equations of simple waves on inviscid shear flows discovered by Blythe, Kazakia & Varley (1972). Such solutions all have propagation speeds proportional to the square root of the depth and contain the uniform flow solution as a special case.

1. Introduction

In the preceding paper, Blythe, Kazakia & Varley (1972, referred to as I) have shown how it is possible to introduce generalized Riemann relations to discuss the occurrence of simple waves on inviscid shear flows in shallow channels. It is somewhat surprising that this problem has not been considered earlier since the results obtained are very similar to those for simple waves on non-sheared flows familiar in many branches of fluid mechanics. The secret of the method is to substitute directly into the equations of motion an expression of the simple wave form and not attempt, as is common in most textbook treatments of the simple wave, to produce a general set of equations in terms of two Riemann invariants.

In this note, the problem will be treated differently from I to show several facets of the theory which may prove instructive to the reader although the fundamental idea is, of course, the same. The main purposes of this presentation is to discuss certain similarity solutions of the governing equations associated with particular forms of shear.

2. Equations of motion

The basic equations of motion for the flow are the unsteady two-dimensional inviscid flow equations in the so-called 'hydraulic approximation':

$$u_t + uu_x + vu_y + gh_x = 0, \tag{2.1}$$

$$u_x + v_y = 0, \tag{2.2}$$

where h is the depth of water in the channel, x and y are Cartesian co-ordinates measured along and perpendicular to the uniform bottom and u and v the associated velocity components. The hydraulic approximation replaces the pressure term in (2.1) by the uniform gravitational pressure ρgh, where ρ is the density and g the acceleration due to gravity. It may be obtained from the full equations by scaling the variables in such a way that lateral variations are small compared with vertical variations. It implies the shallow-water approximation $h_0/L \ll 1$, where h_0 is a characteristic depth and L a typical wavelength.

The surface boundary conditions become

$$v = h_t + uh_x, \tag{2.3}$$

$$u_t + uu_x + gh_x = 0 \tag{2.4}$$

on $y = h$.

At the bottom,

$$v = 0 \quad \text{on} \quad y = 0. \tag{2.5}$$

Simple wave solutions in which the wave speed c depends only on the depth h are sought. Thus

$$u = u(x-ct, y), \tag{2.6}$$

$$h = h(x-ct), \tag{2.7}$$

where $c = c(h)$ only.

Differentiation (2.6) and (2.7), we obtain

$$h_t + ch_x = 0, \tag{2.8}$$

$$u_t + cu_x = 0. \tag{2.9}$$

Equation (2.1) now becomes

$$(u-c)\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial h}{\partial x} = 0. \tag{2.10}$$

Putting $\xi = x - ct$, this equation together with (2.2) may be written as

$$(u-c)\frac{\partial u}{\partial \xi} + w\frac{\partial u}{\partial y} + g\frac{dh}{d\xi} = 0, \tag{2.11}$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial w}{\partial y} = 0, \tag{2.12}$$

where $w = v(\partial\xi/\partial x)^{-1}$.

Further, the boundary conditions are

$$w = (u-c)\,dh/d\xi \tag{2.13}$$

and

$$(u-c)\frac{\partial u}{\partial \xi} + g\frac{dh}{d\xi} = 0 \tag{2.14}$$

on $y = h$, with $w = 0$ on $y = 0$.

Since $h = h(\xi)$, these equations may be written in terms of the independent variables y and h:

$$(u-c)\frac{\partial u}{\partial h} + w_1\frac{\partial u}{\partial y} + g = 0, \tag{2.15}$$

$$\frac{\partial u}{\partial h} + \frac{\partial w_1}{\partial y} = 0, \tag{2.16}$$

with boundary conditions

$$w_1 = u-c, \quad (u-c)\,\partial u/\partial h + g = 0 \quad \text{on} \quad y = h, \tag{2.17}$$

$$w_1 = 0 \quad \text{on} \quad y = 0, \tag{2.18}$$

where

$$w_1 = w\frac{d\xi}{dh} = v\frac{d\xi/dh}{\partial\xi/\partial x} = \frac{v}{\partial h/\partial x}. \tag{2.19}$$

Eliminating $\partial u/\partial h$ from (2.15) and (2.6), we may write

$$\frac{\partial}{\partial y}\left(\frac{w_1}{u-c}\right) = \frac{g}{(u-c)^2}. \tag{2.20}$$

On integrating, we have

$$w_1 = g(u-c)\int_0^y \frac{dy}{(u-c)^2}. \tag{2.21}$$

Evaluation at the upper limit requires, from (2.17),

$$\int_0^h \frac{g\,dy}{(u-c)^2} = 1. \tag{2.22}$$

This result may be compared with the result of Burns (1953) for linearized theory, where u is known explicitly as the undisturbed shear and hence equation (2.22) gives the speed c directly.

Substituting (2.21) into (2.15) gives

$$\frac{\partial u}{\partial h} + \frac{g}{u-c} + g\frac{\partial u}{\partial y}\int_0^y \frac{dy}{(u-c)^2} = 0. \tag{2.23}$$

Eliminating the explicit appearance of u by writing

$$I = g\int_0^y \frac{dy}{(u-c)^2} \tag{2.24}$$

then gives

$$\frac{\partial^2 I}{\partial y\,\partial h} + I\frac{\partial^2 I}{\partial y^2} = 2\frac{\partial I}{\partial y}\left[\frac{\partial I}{\partial y} - \frac{c'}{g^{\frac{1}{2}}}\left(\frac{\partial I}{\partial y}\right)^{\frac{1}{2}}\right], \tag{2.25}$$

where $c' = dc/dh$.

Equation (2.25) is a quasi-linear hyperbolic equation for I provided that the function $c(h)$ is known. The boundary conditions on I are

$$I(h,h) = 1, \quad I(h,0) = 0 \tag{2.26}$$

and $I(h_0, y)$ given by initial conditions posed at some $h = h_0$.

A more convenient form for (2.25) is obtained by introducing the co-ordinate $Y = y/h$ as

$$h\frac{\partial^2 I}{\partial Y\,\partial h} + (I-Y)\frac{\partial^2 I}{\partial Y^2} = \frac{\partial I}{\partial Y} + \frac{2\partial I}{\partial Y}\left[\frac{\partial I}{\partial Y} - \frac{c'h^{\frac{1}{2}}}{g^{\frac{1}{2}}}\left(\frac{\partial I}{\partial Y}\right)^{\frac{1}{2}}\right], \tag{2.27}$$

with $I\ (h, I) = 1$, $I(h, 0) = 0$ and $I(h_0, Y)$ given.

3. Characteristic equations and linearized theory

Equation (2.27) has characteristics

$$(dh)^2(I-Y) - h\,dY\,dh = 0 \tag{3.1}$$

or

$$dh = 0, \quad dY/dh = (I-Y)/h.$$

Along lines h = constant, we can write

$$\begin{aligned} h\,dp + (I-Y)\,dq &= [2q^2 - (2c'h^{\frac{1}{2}}/g^{\frac{1}{2}})\,q^{\frac{3}{2}} + q]\,dY \\ &= 2q[q^{\frac{1}{2}} - a]\,[q^{\frac{1}{2}} - b]\,dY, \end{aligned} \tag{3.2}$$

where a and b are the roots of $\lambda^2-(c'h^{\frac{1}{2}}/g^{\frac{1}{2}})\lambda+1 = 0$ and $p = \partial I/\partial h$, $q = \partial I/\partial Y$. On $(I-Y)\,dh = h\,dY$,

$$dq = 2q\,[q^{\frac{1}{2}}-a]\,[q^{\frac{1}{2}}-b]\,dh. \tag{3.3}$$

The boundary conditions $I = 1$ on $Y = 1$, $I = 0$ on $Y = 0$ and $I = I(y)$ on $h = h_0$ are all prescribed along characteristics. Obviously, there are too many boundary conditions for the usual characteristic-value problem but this is necessary to determine the unknown function $c(h)$ in the differential equation.

The characteristics of (3.3) are, in fact, the particle paths of the motion, as may be demonstrated by evaluating

$$\frac{dy}{dx} = \frac{v}{u-c} = \frac{w_1}{u-c}\frac{\partial h}{\partial x},$$

whence

$$\frac{dh}{dy} = \frac{w_1}{u-c} = I. \tag{3.4}$$

A linearized theory which corresponds to that derived by Burns (1953) may be obtained by expanding in terms of a small parameter ϵ as follows:

$$I = I_0(Y)+\epsilon I_1(Y,(h-h_0)/\epsilon)+\ldots, \tag{3.5}$$

with

$$h = h_0+\epsilon h_1+\ldots$$

and

$$c = c_0+\epsilon c_1+\ldots.$$

Substituting in (2.25) we obtain

$$I_0 = gh_0\int_0^Y \frac{dY}{(u-c_0)^2}$$

and

$$I_1(Y) = \frac{2h_1}{h_0}\int_0^Y \frac{d}{dY}\left[\frac{I_0}{(dI_0/dY)^{\frac{1}{2}}}\right]\left(\frac{dI_0}{dY}\right)^{\frac{3}{2}} dY+I_0(Y)\frac{Yh_1}{h_0}, \tag{3.6}$$

which checks with Burn's theory directly.

4. Similarity solutions

Solutions to (2.27) which are independent of h may be sought. An alternative approach is to expand (2.27) in powers of h with coefficient functions only of Y. The first term depends only on Y and satisfies the equation

$$(I-Y)\frac{d^2I}{\partial Y^2} = \frac{dI}{\partial Y}+2\frac{dI}{dY}\left[\frac{dI}{dY}-\frac{c'h^{\frac{1}{2}}}{g^{\frac{1}{2}}}\left(\frac{dI}{dY}\right)^{\frac{1}{2}}\right], \tag{4.1}$$

where $c'h^{\frac{1}{2}}/g^{\frac{1}{2}}$ must be a constant.

Writing $c'h^{\frac{1}{2}}/g^{\frac{1}{2}} = \alpha$, we obtain $c = 2\alpha(gh)^{\frac{1}{2}}$. Thus all the solutions have propagation speed proportional to $(gh)^{\frac{1}{2}}$. The special case $\alpha = \frac{3}{2}$ or $c = 3(gh)^{\frac{1}{2}}$ corresponds to the no-shear solutions, which can be treated by classical methods.

The solution of the (4.1) subject to the boundary conditions

$$I(1) = 1, \quad I(0) = 0$$

is tedious but not difficult. Writing $J = I-Y$ we obtain

$$JJ'dJ'/dJ = 2(1+J')^2-2\alpha(1+J')^{\frac{3}{2}}+(1+J'), \tag{4.2}$$

where $c'h^{\frac{1}{2}}/g^{\frac{1}{2}} = \alpha$. After a further substitution $N^2 = J' + 1$ this may be integrated to give

$$J = C\frac{(a-N)^{(a^2-1)/a(a-b)}\,(N-b)^{(b^2-1)/b(b-a)}}{N^{1/ab}}, \tag{4.3}$$

where a and b are the roots of $\lambda^2 - \alpha\lambda + \frac{1}{2} = 0$ and C is an arbitrary constant. It will be observed that the boundary conditions on J are $J(1) = 0$, $J(0) = 0$. Thus (4.3) satisfies these conditions at $N = a$ or $N = b$ provided that $a > 1$.

Since $N^2 - 1 = dJ/dY$, we deduce that

$$Y = -C\int\frac{(a-N)^{-1/(2a^2-1)}\,(N-\frac{1}{2}a^{-1})^{(2a^2)/(2a^2-1)}}{N^3}\,dN. \tag{4.4}$$

The variable N is directly related to the velocity y by the relation

$$N^2 = \frac{dI}{dY} = \frac{gh}{(u-c)^2}, \tag{4.5}$$

from which

$$u = (2\alpha - N^{-1})\,(gh)^{\frac{1}{2}}. \tag{4.6}$$

Writing $U = u/(gh)^{\frac{1}{2}}$ we see that

$$U = 1/a \quad \text{at} \quad Y = 0 \tag{4.7}$$

and

$$U = 2a \quad \text{at} \quad Y = 1.$$

Thus (4.4) becomes after a little manipulation

$$Y = \frac{\int_0^{U-1/a} z^{2a^2/(2a^2-1)}\,(2a-a^{-1}-z)^{-1/(2a^2-1)}\,dz}{\int_0^{2a-1/a} z^{2a^2/(2a^2-1)}\,(2a-a^{-1}-z)^{-1/(2a^2-1)}\,dz}. \tag{4.8}$$

On introducing standard notation for the β function we obtain

$$Y = \beta\left(\frac{4a^2-1}{2a^2-1}, \frac{2(a^2-1)}{2a^2-1}, \frac{U-a^{-1}}{2a-a^{-1}}\right)\bigg/\beta\left(\frac{4a^2-1}{2a^2-1}, \frac{2(a^2-1)}{2a^2-1}, 1\right). \tag{4.9}$$

In fact, the function (4.9) is tabulated.

The velocity ranges from $u = (gh)^{\frac{1}{2}}/a$ at $y = 0$ to $u = 2a(gh)^{\frac{1}{2}}$ at $y = h$ and the propagation speed $c = (2a + a^{-1})\,(gh)^{\frac{1}{2}}$. We note that for $a > 1$, $c > 2a > u_{\max}$. Thus $u \neq c$ anywhere in the range and there is no critical point. Since $J(1) = 0$ implies $I(1) = 1$ the velocity profile (4.8) satisfies the Burns condition given by equation (2.2). A theorem given by Burns (1953) states that, for the linearized theory, a velocity profile of the form (4.8) should give two propagation speeds c_1 and c_2 such that $c_1 > U(1) > U(0) > c_2$. By inserting the profile (4.8) in the condition (2.2) it can be shown using elementary theory of the hypergeometric function that

$$I(1) = 1 - \frac{(c-a^{-1})^{1/(2a^2-1)}\,[c-(2a+a^{-1})]}{(c-2a)^{2a^2/(2a^2-1)}}. \tag{4.10}$$

Thus the Burns condition gives $c_1 = 2a + a^{-1}$ and $c_2 = 1/a = U(0)$. The bottom is thus a critical point of the second propagation speed. It will be observed that

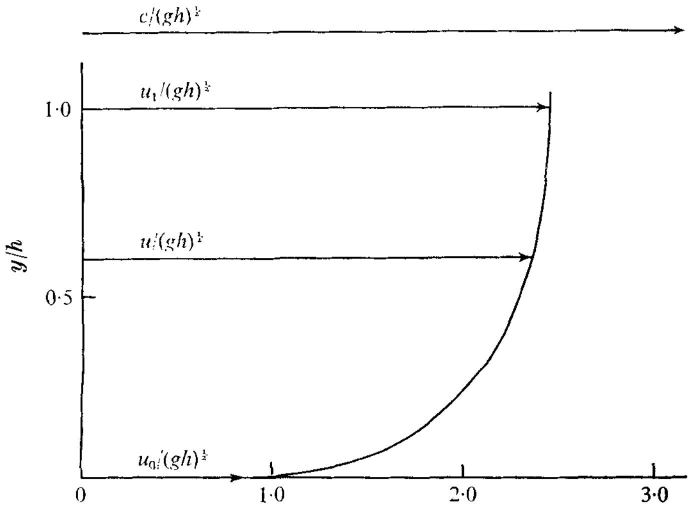

FIGURE 1. Velocity profile for $c = 3{\cdot}265\,(gh)^{\frac{1}{2}}$.

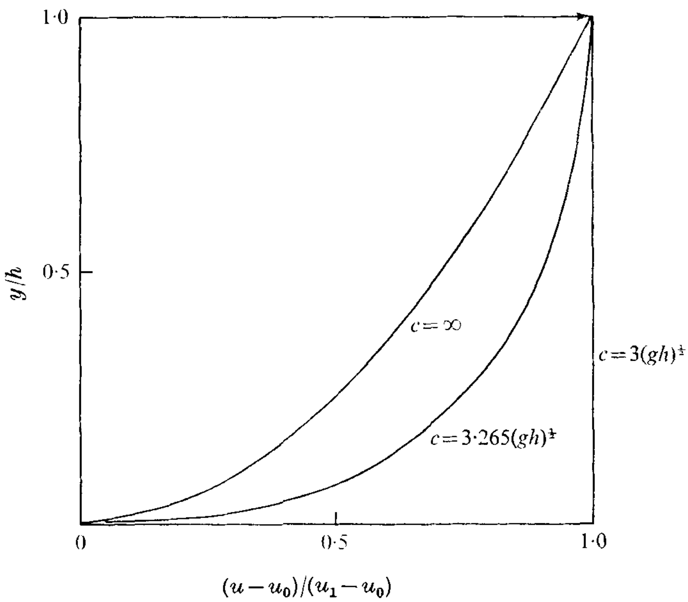

FIGURE 2. Velocity profiles for $c = 3{\cdot}0\,(gh)^{\frac{1}{2}}$, $3{\cdot}265\,(gh)^{\frac{1}{2}}$ and ∞.

$I(Y)$ still converges for $c = c_2$ and Burns' theorem must be modified to include this case. In the nonlinear theory $I(Y)$ with the profile (4.8) and $c = c_2$ does not satisfy the differential equation (4.1). We can only assume that this profile with $c = 1/a$ does not develop as a similarity solution.

The special case $a = 1$, $\alpha = \frac{3}{2}$ gives $u = 2(gh)^{\frac{1}{2}}$ on $y = h$ and $c = 3(gh)^{\frac{1}{2}}$, and this solution corresponds to no shear.

A typical profile for $a^2 = 1{\cdot}5$ and $c = 3{\cdot}265\ (gh)^{\frac{1}{2}}$ is shown in figure 1. It will be seen that (4.9) gives Y as a function of $(u-u_0)/(u_1-u_0)$, where u_0 is velocity at $Y = 0$ and u_1 is velocity at $Y = 1$. By plotting the result in terms of these variables, the whole range of a from unity to infinity can be shown (figure 2).

5. Conclusions

The similarity solutions derived in §4 form a set of solutions which are non-critical in the sense of I in that they do not possess a critical layer at which $u = c$. They are a more general class of solutions than the simple waves of classical shallow-water theory without shear, which themselves belong to the class.

The author is indebted to Professor P. A. Blythe, Professor P. H. Roberts, Dr P. Baldwin and Dr R. S. Johnson for many helpful comments and suggestions.

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